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Prediction of cumulative fatigue damage

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The life of a component subjected to a constant stress can be determined using the *S–N* diagram. Prediction of cumulative fatigue damage or fatigue life of a component subjected to a given stress sequence of varying amplitude is difficult, if not impossible. Miner's rule and Kramer's equation can predict cumulative fatigue damage with a reasonable degree of accuracy for limited cases. The development of Kramer's equation was studied and a modification suggested. Cumulative fatigue data generated using aluminium alloys 2011-T3 and 2024-T4, were analysed using Miner's, Kramer's and the modified equations. While Miner's rules and Kramer's equation predict cumulative fatigue damage with tolerable error for a few cases, the modified equation has shown close agreement between the experimental and theoretical values of fatigue life or cumulative fatigue damage for the materials studied in this investigation.

1. Introduction

Fatigue loads carried by structural components and machine parts are rarely of constant magnitude. It is not uncommon for a part to be subjected to multilevel fluctuating stresses of high frequency.

Prediction of cumulative fatigue damage for a given stress-cycle sequence or the life of a component for a set of stress levels is difficult, if not impossible. The *S–N* diagram, which is a plot of fatigue strength against the number of cycles to failure, by itself, does not provide this information without some analysis.

Several investigators have studied the phenomena of cumulative damage and proposed mechanisms to determine the damage accumulated in a part subjected to fatigue stresses of varying amplitude [1–9], but none of these can predict the fatigue failure with any accuracy. Most recently, Kramer [10] has proposed a linear cumulative damage rule similar to that of Miner [1] with one major difference in that the former includes the damage histories of the previous stages in its terms.

Extensive experimental fatigue data were generated using aluminium alloys 2011-T3 and 2024-T4 for stress ratios of $R = -1, -0.5$, and 0 for various stress sequences and Kramer's equation was used to predict the cumulative damage [15–19]. It was found that for aluminium alloy 2011-T3, the predicted cumulative fatigue damage and fatigue life for low–high and low–high–mixed stress sequences, for a stress ratio of -1, were in close agreement with those obtained experimentally [20]. For most of the other cases, Kramer's equation predicted very conservative values. For aluminium alloy 2024-T4, the predicted cumulative fatigue damage and fatigue life for low–high and low–high–mixed stress sequences, for stress ratios of -1 and -0.5 were in close agreement with those obtained experimentally [21]. For all other cases, Kramer's equation predicted very conservative values.

The purpose of this part of the investigation was to examine the development of Kramer's equation and modify it so that it can predict cumulative fatigue damage accurately for all stress ratios for various stress sequences.

2. Modified Kramer's equation

For the prediction of cumulative fatigue damage both Miner's rule and Kramer's theory use data from the *S–N* diagram but it is different.

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Miner assumed that the phenomenon of cumulative fatigue damage under repeated loads was related to the net work absorbed by the materials. The total amount of work that can be absorbed produces failure (under the further assumption that no work-hardening occurs). The number of loading cycles applied expressed as a percentage of the number to a failure at a given stress level would be the proportion of useful life expended. If W represents the net work absorbed at failure, then

$$\frac{W_i}{W} = \frac{n_i}{N_i}, \tag{1}$$

where W_i represents the work absorbed when the material is stressed at σ_i for n_i cycles, and N_i is the total number of cycles required for the material to fail when stressed at σ_i.

For failure to occur

$$W_1 + W_2 + W_3 \ldots = W,$$

or

$$\frac{W_1}{W} + \frac{W_2}{W} + \frac{W_3}{W} = 1. \tag{2}$$

Substituting Equation 1 into Equation 2,

$$n_i/N_i = 1 \tag{3}$$

On the other hand, Kramer believed that work done during cycling on the specimen is primarily used to work harden the surface layer [10–14]. He defined the increase in proportional limit as surface layer stress (σ_s). When this surface layer stress reached a critical value (σ_s^*), failure occurred. The critical surface layer stress is independent of the amplitude of the stress applied.

On the basis of this argument, Kramer has developed his equation for predicting cumulative fatigue damage:

$$\sigma_1^P n_1 + \sigma_2^P n_2 \left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_1} + \sigma_3^P n_3 \left(\frac{\sigma_2}{\sigma_3}\right)^{Pf_2} \left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_2 f_1} + \ldots = \frac{\sigma_s^*}{\alpha} = \beta \tag{4}$$

where $\sigma_1, \sigma_2, \ldots$ are the applied stresses, $n_1, n_2, \ldots$ are the numbers of cycles applied at stress $\sigma_1, \sigma_2, \ldots$ respectively, m is the slope of the S–N curve which is often of the form $Y = CX^m$, $P = (-1/m)$ is a material constant, $\sigma_s^*$is the critical surface layer stress, $\beta = C^P$ is a material constant and

$$f_1 = \sigma_1^P n_1,$$

$$f_2 = \sigma_2^P n_2 \left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_1}, \text{ etc},$$

are the damage histories in the previous stages of the stress cycle loading process.

The equation can also be expressed as

$$\frac{\sigma_1^P n_1}{\beta} + \frac{\sigma_2^P n_2}{\beta}\left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_1} + \frac{\sigma_3^P n_3}{\beta}\left(\frac{\sigma_2}{\sigma_3}\right)^{Pf_2}\left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_2 f_1} + \ldots = 1 \tag{5}$$

which means that, when the cumulative fatigue damage for all the stages equals unity, failure will ocurr.

There exists a contrast between Miner's rule and Kramer's theory. The former assumes that there is total work absorption during fatiguing while the latter believes that energy is used primarily for work hardening of the surface layer. Miner's law also does not take into account the residual stresses, while Kramer ignores the cumulative effect of cyclic ratios; he considers cumulative effect only on applied stress levels.

From these two theories it looks logical to assume that part of the energy during fatiguing is consumed to work harden the surface layer and part of it is absorbed by the material. On the basis of this assumption a modified version of Kramer's equation is proposed:

$$\frac{n_1\sigma_1^P}{\beta} + \frac{n_2\sigma_2^P}{\beta}\left(\frac{n_1}{n_2}\right)^{f_1}(\sigma_1/\sigma_2)^{Pf_1} + \frac{n_3\sigma_3^P}{\beta}\left(\frac{n_2}{n_3}\right)^{f_2}\left(\frac{n_1}{n_2}\right)^{f_1 f_2}\left(\frac{\sigma_2}{\sigma_3}\right)^{Pf_2}\left(\frac{\sigma_1}{\sigma_2}\right)^{Pf_1 f_2} + \ldots = 1 \tag{6}$$

Details concerning the development of this equation may be found elsewhere [18, 22].

It may be pointed out that the theoretical value of cumulative fatigue damage for any material is unity. Determination of cumulative fatigue damage consists of determining the numerical value of the expression on the left side of Equation 6. Prediction of fatigue damage or life consists of determining the fatigue life for a given stress when the material has already been fatigued at one or more stress levels without causing failure.

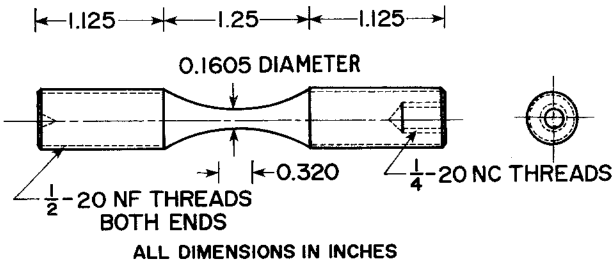

Figure 1 Fatigue specimen.

3. Experimental work

3.1. Specimen materials

In this investigation specimens of 2011-T3 and 2024-T4 aluminium alloys were used for fatigue tests. Chemical composition, mechanical and physical properties of the alloys are given elsewhere [18, 22]. The properties that make these alloys suitable for a wide variety of uses are appearance, fabricability, physical and mechanical properties, corrosion, resistance and machinability, etc. The standard temper designation system for aluminium alloys consists of a letter indicating the basic temper. T3 indicates solution heat treated and cold worked, naturally aged to stable condition. T4 indicates solution heat treated and naturally aged to stable condition.

3.2. Specimen preparation

The fatigue specimens used in this investigation were made from $\frac{1}{2}$in. diameter rod. The design of the specimen is shown in Fig. 1. The length-to-diameter ratio of the gauge section of the specimen was chosen 2 to 1 to avoid buckling. The threaded ends were turned on a lathe while the profile between the threaded portions was formed on a Tensilkut-Tensilath. After the specimen had been machined, the gauge section was mechanically polished using grade 800 silicon carbide paper. Finally, the specimen was electropolished to obtain a surface free of tool marks and other irregularities.

3.3. Test procedure

The fatigue machine used in this investigation is equipped with an automatic hydraulic load maintainer which adjusts the preload continuously to a pre-set value without affecting the cyclical load. The cyclical load can be adjusted manually up to ± 600 lbf (2670 N). The test frequency of the machine ranges from 600 to 2200 cycle min^{-1}. A load cell is provided to read the direct tensile or compressive load on the specimen. The load cell is connected to a strain indicator which is calibrated to read the stress in the specimen directly.

The testing of each specimen was done in four different stages. For the first stage to the third stage, the load was set and the number of cycles applied at a particular load was predetermined. In the last (fourth) stage, after the load had been set, the test was allowed to continue until the specimen failed.

The specimens of 2011-T3 aluminium alloy were tested under three different stress ratios of $R = -1$, -0.5 and 0, while the specimens of 2024-T4 aluminium alloy were fatigued under $R = -1$ and -0.5. Fig. 2 explains the stress ratios.

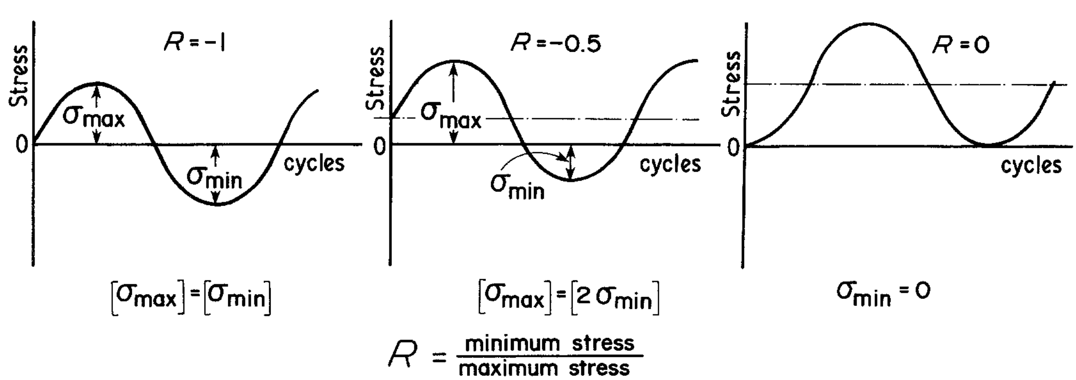

Figure 2 Stress ratios.

TABLE I Cumulative fatigue data for a stress ratio of −0.5, high–low stress sequences (material, aluminium alloy 2011-T3)

Specimen Number	Maximum stress								D_T		
	Stage 1		Stage 2		Stage 3		Stage 4		Kramer	Miner	Modified Kramer
	Stress (10^3 psi)*	Number of cycles	Stress (10^3 psi)	Number of cycles	Stress (10^3 psi)	Number of cycles	Stress (10^3 psi)	Number of cycles			
1	40	4000	35	8000	30	30 000	25	58 500	1.554	1.139	1.116
2	40	4000	35	8000	30	30 000	25	58 700	1.239	0.972	0.852
3	40	5000	35	7000	30	35 000	25	37 800	1.432	1.131	1.153
4	40	5000	35	7000	30	35 000	25	60 100	1.811	1.238	1.233
								Average damage	1.509	1.120	1.088

*10^3 psi ≡ 6.89 N mm^{-2}.

4. Results and discussion

In this paper the results of the analysis of the data generated on aluminium alloys 2011-T3 and 2024-T4, using Miner's, Kramer's, and Kramer's modified equations, are discussed. The experimental data generated using these alloys may be found elsewhere [15–22]. Approximately 350 specimens were used to generate the *S–N* diagrams and cumulative fatigue data for twenty different stress sequences and stress ratios.

Table I shows the cumulative fatigue data produced using aluminium alloy 2011-T3 for a stress ratio of $R = -0.5$ for the low-to-high stress sequence. For specimens 1 and 2 the maximum stress in the first stage was 40 000 lbf in.$^{-2}$ (276 N mm^{-2}) for 4000 cycles, 35 000 lbf in.$^{-2}$ (241 N mm^{-2}) in the second stage for 8000 cycles, 30 000 lbf in.$^{-2}$ (207 N mm^{-2}) for 30 000 cycles in the third stage and finally 25 000 lbf in^{-2} (172 N mm^{-2}) in the fourth stage until the specimen failed at 58 500 and 58 700, respectively.

For specimens 3 and 4 the stress sequence was 40 000 lbf in.$^{-2}$ for 5000 cycles, 35 000 lbf in.$^{-2}$ for 7000 cycles, 30 000 lbf in.$^{-2}$ for 35 000 cycles and finally 25 000 lbf in.$^{-2}$ for 37 800 and 60 100 cycles, respectively in the fourth stage. The table also shows the cumulative fatigue damage (D_T) calculated in each case using Kramer's, Miner's and Kramer's modified equations. It can be seen that the average value of the total damage determined using Kramer's modified equation is more realistic than those calculated using the other two equations.

Cumulative fatigue data for the other nineteen stress sequences is not presented here to conserve space. Table II shows a comparison of the fatigue damages determined using Kramer's, Miner's and Kramer's modified equations for all the stress sequences used in the investigation. For each stress sequence the average, maximum, and minimum values of the cumulative fatigue damage are shown. It can be seen that the modified equation predicts less conservative fatigue damage and shows less scatter in the values of average damage that those predicted by the other two theories. However, there are cases where Miner's rule and Kramer's equation have yielded better values of cumulative fatigue damage. Miner's rule, for example, showed better average damage values in the following two stress sequences:

1. Aluminium 2024-T4, $R = -0.5$, high–low–mixed,
2. Aluminium 2011-T3, $R = 0$, high–low–mixed.

TABLE II Comparison of cumulative fatigue damage

Material	Stress sequence*	Kramer			Miner			Modified Kramer			No. of specimen	R
		D_{ave}	D_{max}	D_{min}	D_{ave}	D_{max}	D_{min}	D_{ave}	D_{max}	D_{min}		
2024-T4	L–H	1.04	1.20	0.89	1.203	1.315	0.982	1.134	1.277	0.988	10	−1
aluminium	L–HM	1.04	1.30	0.933	1.14	1.26	1.016	1.13	1.252	1.029	8	
alloy	H–L	1.204	1.558	0.855	1.050	1.378	0.79	1.02	1.252	0.863	10	
	H–LM	1.212	1.50	1.074	1.050	1.166	0.929	1.016	1.105	0.941	7	
	L–H	0.956	1.08	0.814	1.097	1.208	0.915	1.10	1.221	0.953	10	−0.5
	L–HM	1.013	1.070	0.977	1.175	1.271	1.106	1.11	1.188	1.057	6	
	H–L	1.291	1.423	1.180	1.060	1.151	0.953	1.006	1.047	0.982	7	
	H–LM	1.136	1.309	1.048	0.989	1.121	0.913	0.961	1.034	0.924	7	
2011-T3	L–H	0.982	1.02	0.881	1.253	1.274	1.011	1.220	1.274	1.044	18	−1
aluminium	L–HM	1.008	1.045	0.971	1.162	1.250	1.089	1.187	1.288	1.060	18	
alloy	H–L	2.340	2.859	1.596	1.511	1.717	1.252	1.426	1.687	1.138	18	
	H–LM	2.466	2.673	2.174	1.851	1.942	1.771	1.785	1.933	1.582	18	
	L–H	0.718	0.816	0.674	0.858	0.971	0.799	0.931	1.077	0.878	5	−0.5
	L–HM	0.696	0.817	0.499	0.761	0.866	0.581	0.806	0.875	0.659	5	
	H–L	1.509	1.811	1.239	1.120	1.238	0.972	1.088	1.233	0.852	4	
	H–LM	1.503	1.536	1.419	1.235	1.281	1.149	1.214	1.248	1.176	4	
	L–H	1.365	1.486	1.286	1.519	1.681	1.407	1.296	1.359	1.254	4	0
	L–HM	1.578	1.686	1.532	1.957	2.114	1.887	1.573	1.676	1.519	4	
	H–L	1.375	1.471	1.245	1.130	1.206	1.031	1.078	1.113	1.063	4	
	H–LM	1.479	1.495	1.445	1.197	1.231	1.169	1.215	1.237	1.176	4	

*L–H = low–high; L–HM = low–high mix; H–L = high–low; H–LM = high–low mix.

Kramer's equation showed better results in the following six stress sequences:

1. Aluminium 2024-T4, $R = -1$, low–high,
2. Aluminium 2024-T4, $R = -1$, low–high–mixed,
3. Aluminium 2024-T4, $T = -0.5$, low–high,
4. Aluminium 2024-T4, $R = -0.5$, low–high–mixed,
5. Aluminium 2011-T3, $R = -1$, low–high,
6. Aluminium 2011-T3, $R = -1$, low–high–mixed.

In the rest of the twelve cases, the modified equation has predicted better (close to unity) values of cumulative fatigue damage.

5. Conclusions

Miner's equation for the prediction of cumulative fatigue damage was developed based on the assumption that the work done during fatiguing was completely absorbed by the material. Kramer, on the other hand, assumed that the entire work was used to work harden the surface layer.

A modified Kramer's equation was developed based on the argument that a part of the work done during fatiguing was absorbed by the material while the rest was used in work hardening the surface layer.

Analysis of the experimental data generated using aluminium alloys 2011-T3 and 2024-T4 for stress ratios of $R = -1$, -0.5 and 0 for low–high, low–high–mixed, high–low, and high–low–mixed sequences has lead to the conclusion that the modified Kramer's equation shows a close agreement between the experimental and theoretical values of cumulative fatigue damage and fatigue life for most of the stress sequences used in this study.

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